Strong Compactness, Easton Functions, and Indestructibility

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The work I am about to discuss is motivated by conversations I had with Stamatis Dimopoulos in late October 2017, who essentially asked me the following

Question: Is it possible to realize suitable Easton functions while preserving the strong compactness of a non-supercompact strongly compact cardinal?

We will say F is an *Easton function for the model M of ZFC* if F satisfies the following conditions:

1. Either $F \in M$ (if F is a set) or F is definable over M (if F is a proper class).

2. $\mathrm{dom}(F)$ is a class of M-regular cardinals.

3. $\mathrm{rge}(F)$ is a class of M-cardinals.

4. For every $\kappa \in \mathrm{dom}(F)$, $F(\kappa) > \kappa$.

5. If $\kappa < \lambda$, $\kappa, \lambda \in \mathrm{dom}(F)$, $F(\kappa) \leq F(\lambda)$.

6. For every $\kappa \in \mathrm{dom}(F)$, $\mathrm{cf}(F(\kappa)) > \kappa$.

A model M^* of ZFC *realizes an Easton function* F if $M^* \models$ "For every regular cardinal δ, $2^\delta = F(\delta)$".

If we are willing to force to change a supercompact cardinal κ into a non-supercompact strongly compact cardinal, then it is possible to construct without too much difficulty a model in which a suitable ground model Easton function F has been realized.

For instance, let F be an Easton function definable by a Δ_2 formula in a model V of ZFC + GCH in which κ is supercompact. A theorem of Menas shows that it is possible to force over V to obtain a model V_1 in which κ remains supercompact and the Easton function F has been realized.

Suppose we now force over V_1 with the Magidor iteration of Prikry forcing which adds a cofinal ω sequence to each measurable cardinal $\delta < \kappa$. In the resulting model V_2, κ is both the least strongly compact and least measurable cardinal, so in particular, κ is not 2^κ-supercompact. In addition, since the Magidor iteration preserves both cardinals and the size of power sets, in V_2, F is still realized on the regular cardinals in the same way as it was in V_1.

In a similar vein, suppose that F is an Easton function definable by a Δ_2 formula in a model V of ZFC + GCH in which λ is a supercompact limit of supercompact cardinals. The aforementioned theorem of Menas shows that it is possible to force over V to obtain a model V_1 in which λ remains a supercompact limit of supercompact cardinals and the Easton function F has been realized.

In V_1, let $\kappa < \lambda$ be the least measurable limit of supercompact cardinals. Another theorem of Menas shows that in V_1, κ is both strongly compact and not 2^κ-supercompact. (Menas' theorem actually states that ZFC $\vdash$ "The least cardinal δ which is both measurable and a limit of supercompact cardinals is strongly compact and not 2^δ-supercompact".) In particular, by starting with hypotheses stronger than the existence of a measurable limit of supercompact cardinals, it is possible to force and construct a model containing a non-supercompact strongly compact cardinal in which F has been realized.

Our goal, however, is to start with a model V of ZFC containing a particular strongly compact cardinal κ which isn't supercompact and a suitable Easton function F. We would then like to force over V, preserve the fact that κ is the same sort of non-supercompact strongly compact cardinal, and realize the Easton function F. (So roughly speaking, we don't wish to use more strong compactness or supercompactness assumptions than are prima facie necessary.)

Although it is still unknown how to do this in complete generality, the following theorem and its immediate corollary, which will be the main focus of the rest of this lecture, provide a partial answer.

Theorem 1 *Suppose $V \models$ "ZFC + GCH + κ is the least measurable limit of supercompact cardinals". There is then a partial ordering $\mathbb{P} \in V$ having cardinality κ such that:*

1. *$V^{\mathbb{P}} \models$ "ZFC + κ is the least measurable limit of supercompact cardinals + $2^\delta = \delta^+$ for every $\delta \geq \kappa$".*

2. *In $V^{\mathbb{P}}$, every supercompact cardinal $\delta < \kappa$ has its supercompactness indestructible under δ-directed closed forcing.*

3. *In $V^{\mathbb{P}}$, for any regular cardinal $\lambda > \kappa$, there is a κ-directed closed cofinality preserving partial ordering $\mathbb{Q}(\lambda)$ such that $V^{\mathbb{P} * \dot{\mathbb{Q}}(\lambda)} \models$ "$2^\delta = \lambda$ for every $\delta \in [\kappa, \lambda)$ + κ is measurable".*

Theorem 1 tells us that starting with a model V of ZFC + GCH in which κ is the least measurable limit of supercompact cardinals, it is possible to force over V to blow up the size of κ's power set to be any regular cardinal $\lambda > \kappa$ while preserving the fact that κ is the least measurable limit of supercompact cardinals. This provides an example of a non-supercompact strongly compact cardinal the size of whose power set can be made arbitrarily large without the use of additional strong compactness or supercompactness assumptions.

Corollary 1 *Suppose F is an Easton function for $V^{\mathbb{P}}$ with domain $A = \{\delta \mid \delta \geq \kappa$ is a regular cardinal$\}$ such that $F(\kappa)$ is regular and $F(\delta) = F(\kappa)$ for every $\delta \in [\kappa, F(\kappa))$. Then it is possible to realize F over $V^{\mathbb{P}}$ while preserving that κ is the least measurable limit of supercompact cardinals.*

To prove Corollary 1 from Theorem 1, let F be the Easton function for $V^{\mathbb{P}} = V_1$ satisfying the above conditions. In $V_1^{\mathbb{Q}(F(\kappa))}$, let $\mathbb{R}$ be the usual Easton product of Cohen posets which realize $F \restriction \{\delta \mid \delta \geq F(\kappa)$ is a regular cardinal$\}$. Then because $V_1^{\mathbb{Q}(F(\kappa))} \models$ "$\mathbb{R}$ is $F(\kappa)$-directed closed", F is realized in $V_1^{\mathbb{Q}(F(\kappa)) * \dot{\mathbb{R}}}$. Since $V_1 \models$ "Each supercompact cardinal $\delta < \kappa$ is indestructible under δ-directed closed forcing and $\mathbb{Q}(F(\kappa)) * \dot{\mathbb{R}}$ is κ-directed closed", $V_1^{\mathbb{Q}(F(\kappa)) * \dot{\mathbb{R}}} \models$ "κ is the least measurable limit of supercompact cardinals". This completes the proof of Corollary 1. $\square$

Some remarks:

1. If we start with hypotheses stronger than the existence of a measurable limit of supercompact cardinals, then we can establish an improved version of Corollary 1.

Specifically, Hamkins has shown that it is possible to force over a model V containing a supercompact limit of supercompact cardinals to obtain a model V_1 in which the least measurable limit of supercompact cardinals κ is such that $2^\delta = \delta^+$ for every $\delta \geq \kappa$, every supercompact cardinal $\delta < \kappa$ has its supercompactness indestructible under δ-directed closed forcing, and κ's measurability is indestructible under *arbitrary* κ-directed closed forcing. Take V_1 as our ground model.

Suppose F is now *any* Easton function defined on the regular cardinals greater than or equal to κ and $\mathbb{Q}$ is the usual Easton poset realizing F over V_1. Then since $V_1 \vDash$ "$\mathbb{Q}$ is κ-directed closed", the arguments used in the proof of Corollary 1 show that $\mathbb{Q}$ is as desired. F is realized in $V_1^{\mathbb{Q}}$, where κ remains the least measurable limit of supercompact cardinals.

2. We can also in analogy to the previous remark realize an arbitrary Easton function F having domain greater than or equal to κ for κ an indestructibly supercompact cardinal such that $2^\delta = \delta^+$ for every $\delta \geq \kappa$. Call the resulting model V_1. If we once again force over V_1 with the Magidor iteration of Prikry forcing which adds a cofinal ω sequence to each measurable cardinal $\delta < \kappa$, F will be realized in a model V_2 of ZFC in which κ is both the least measurable and least strongly compact cardinal. As we previously observed, in V_2, κ isn't 2^κ supercompact.

Turning now to the proof of Theorem 1, suppose $V \models$ "ZFC + GCH + κ is the least measurable limit of supercompact cardinals". The proof of Theorem 1 is divided into three parts.

Part I: We force to make each supercompact cardinal $\delta < \kappa$ indestructible under δ-directed closed forcing using an Easton support iteration of length κ. This will preserve GCH at and above κ and also preserve the fact that κ is the least measurable limit of supercompact cardinals. Call the resulting model V_1.

Part II: Force over V_1 using Woodin's notion of fast function forcing $\mathbb{F}_\kappa$ to add a fast function f for κ. ($\mathbb{F}_\kappa$ is defined as $\{p \mid p : \kappa \to \kappa$ is a function such that dom(p) consists of inaccessible cardinals, $|p \restriction \lambda| < \lambda$ for every inaccessible cardinal $\lambda \leq \kappa$, and for every $\delta \in$ dom(p), $p''\delta \subseteq \delta\}$, ordered by inclusion.)

If $V_1 \vDash$ "$\delta < \kappa$ is supercompact", then $f = f \restriction \delta \times f \restriction [\delta, \kappa)$, where $f \restriction [\delta, \kappa)$ is V_1-generic over a poset $\mathbb{F}_{[\delta,\kappa)}$ which is δ-directed closed. Since $V_1 \vDash$ "δ's supercompactness is indestructible under δ-directed closed forcing", $V_1[f \restriction [\delta, \kappa)] \vDash$ "δ is supercompact". Since $f \restriction \delta$ is a fast function for δ when forcing over either V_1 or $V_1[f \restriction [\delta, \kappa)]$, $V_1[f \restriction [\delta, \kappa)][f \restriction \delta] = V_1[f] \vDash$ "δ is supercompact" as well. In $V_1[f]$, κ remains the least measurable limit of supercompact cardinals, and GCH continues to hold at and above κ.

Part III: Force over $V_1[f] = V_2$ with Hamkins' lottery preparation $\mathbb{L}$ defined with respect to f using only suitably directed closed posets to obtain the model V_3. Intuitively, $\mathbb{L}$ may be thought of as a kind of analogue of the Laver preparation defined with respect to f.

More specifically, suppose $\mathcal{A}$ is a collection of posets. The *lottery sum* $\oplus\mathcal{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathcal{A}$ and $p \in \mathbb{P}\} \cup \{0\}\}$, ordered by $\langle \mathbb{P}, p \rangle \le \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. (Intuitively, the generic object over the lottery sum $\oplus\mathcal{A}$ first selects a poset in $\mathcal{A}$ and then forces with it.) Given this definition, $\mathbb{L}$ may now be defined as the Easton support iteration of length κ which does nontrivial forcing only at those stages $\delta < \kappa$ such that $f(\delta)$ is defined and $\delta \le f(\delta)$. At such a δ, we force with the lottery sum in $V^{\mathbb{P}_\delta}$ of all posets in $H(f(\delta)^+)$ which are δ-directed closed.

In V_3, as in Part II above, κ remains the least measurable limit of supercompact cardinals, and GCH continues to hold at and above κ.

We may now use results of Hamkins (found in his papers “The Lottery Preparation”, *APAL* 2000 and “Tall Cardinals”, *MLQ* 2009) to infer the following:

1. In V_3, every supercompact cardinal $\delta < \kappa$ has its supercompactness indestructible under δ-directed closed forcing.

2. Because $V_2 \models$ "κ is strongly compact", $V_2 \models$ "κ is a tall cardinal" as well.

   (κ is a *tall cardinal* means that for every ordinal $\delta > \kappa$, there is an elementary embedding $j_\delta : V \to M$ such that κ is the critical point of j_δ, $j_\delta(\kappa) > \delta$, and $M^\kappa \subseteq M$.)

3. Because $V_2 \models$ "κ is a tall cardinal", we may now apply arguments due to Woodin for forcing the failure of GCH at a measurable cardinal from relatively weak hypotheses. These allow us to infer that for any V_2- or V_3-regular cardinal $\lambda > \kappa$, there is a κ-directed closed cofinality preserving partial ordering $\mathbb{Q}(\lambda) \in V_3$ such that $V_3^{\mathbb{Q}(\lambda)} \models$ "$2^\delta = \lambda$ for every $\delta \in [\kappa, \lambda)$ + κ is measurable".

Parts I – III complete the proof sketch of Theorem 1. $\square$

Note that if we wish the Easton function F of Corollary 1 to be such that $F(\kappa) = \kappa^+$ (i.e., if we don't wish to blow up the size of the power set of κ), then it is only necessary in the proof of Theorem 1 to force with the poset of Part I. (Recall that this partial ordering makes each supercompact cardinal $\delta < \kappa$ indestructible under δ-directed closed forcing, preserves GCH at and above κ, and also preserves the fact that κ is the least measurable limit of supercompact cardinals.) This is since the Easton poset $\mathbb{Q}$ realizing F can be defined to be κ^+-directed closed. Thus, forcing with $\mathbb{Q}$ will preserve that κ is the least measurable limit of supercompact cardinals.

I will conclude with the following questions:

1. Is it possible to remove the restrictions on the Easton function F in the proof of Corollary 1?

2. In general, what sorts of Easton functions can be realized in the presence of non-supercompact strongly compact cardinals?

3. Is it possible to realize suitable Easton functions (e.g., $F(\delta) = \delta^{++}$ for every regular cardinal δ) while preserving all ground model strong and supercompact cardinals?

This last question, while not directly related to the topic of this lecture, seems to be technically connected.

Thank you all very much for your attention!